Equation (9) is confined to σ larger than unity, because larger σ implies smaller drag. According to Eq. (9), one of necessary conditions for σ is given by

$$4/3 \geqq \sigma \geqq 1 \tag{10}$$

Hence, the following inequality holds for all $y \in [0, +1]$:

$$\frac{d\gamma}{dy} = -\frac{(3\sigma - 2)\gamma_{Re} y}{\sigma^3\sqrt{1 - y^2}}\left[1 - \frac{6(\sigma - 1)}{3\sigma - 2} \times \left\{\sqrt{1 - y^2}\,\ell n \left|\frac{1 - \sqrt{1 - y^2}}{1 + \sqrt{1 - y^2}}\right| + 1\right\}\right] \leqq 0 \tag{11}$$

Then we can deduce: 1) $\gamma(y)$ is not negative for all y because of the inequality, Eq. (11), and conditions on $\gamma(y)$; that is, $\gamma(0) > 0$ and $\gamma(1) = 0$; 2) the displacement of the trailing edge $h(y)$ is positive, as far as $\gamma(y)$ is positive (Fig. 1); and 3) Eq. (8) implies that $c(y)$ is positive, because both $\gamma(y)$ and $h(y)$ are positive.

This deduction leads to the conclusion that Eq. (10) is also the sufficient condition of positive $c(y)$. Hence, we obtain the optimum semispan ratio σ_{opt} and the optimum drag ratio

$$\sigma_{\text{opt}} = 4/3 \tag{12}$$

$$D/D_e = 27/32 \tag{13}$$

Optimum Solutions

Substituting Eq. (12) into Eq. (1), we obtain

$$\gamma(y) = \frac{27}{32}\gamma_{Re}\left\{\sqrt{1 - y^2} + \frac{1}{2}y^2\,\ell n\left|\frac{1 - \sqrt{1 - y^2}}{1 + \sqrt{1 - y^2}}\right|\right\} \tag{14}$$

Figure 2 shows the numerical result of this optimum circulation compared with the elliptic loading. Most of the necessary lift is produced near the wing root so that the bending moment would not exceed the given value.

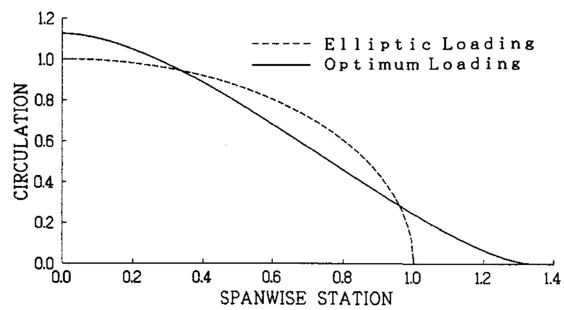

Fig. 2 Circulation distributions normalized by γ_{Re}.

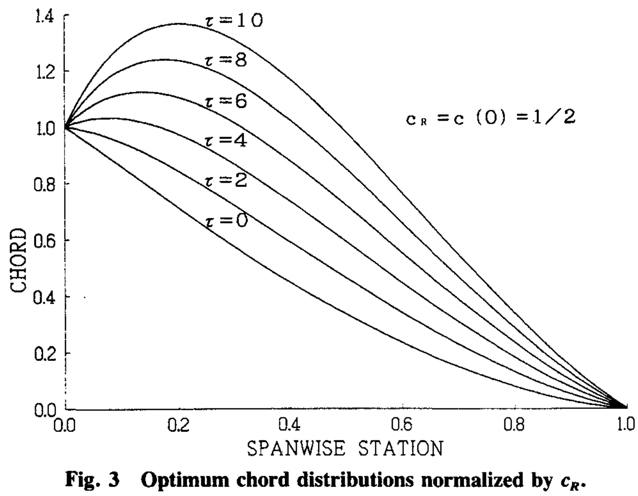

Fig. 3 Optimum chord distributions normalized by c_R.

Substitution of Eq. (12) into Eq. (8) yields

$$c(y)/c(0) = \left[\sqrt{1 - y^2} + \frac{y^2}{2}\,\ell n\left|\frac{1 - \sqrt{1 - y^2}}{1 + \sqrt{1 - y^2}}\right| + \frac{\pi\tau}{12}\left\{1 + (\pi - 1)|y| - \left(1 + \frac{3}{2}y^2\right)\sqrt{1 - y^2} - \frac{y^4}{4}\,\ell n\left|\frac{1 - \sqrt{1 - y^2}}{1 + \sqrt{1 - y^2}}\right| - 2y\sin^{-1}y\right\}\right] \Big/ \left\{1 + \frac{\pi^2}{4}c_R|y|\right\} \tag{15}$$

Figure 3 shows numerical results of Eq. (15), in the case where c_R is $\frac{1}{2}$. Hanggliders have fuller chord distributions than that of a rigid wing (i.e., $\tau = 0$). To realize the optimum circulation distribution (Fig. 2), wing area must increase wherever the effective angle of attack is reduced by the twist due to the displacement of trailing edge.

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Optimization of Constant Altitude–Constant Airspeed Flight of Turbojet Aircraft

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Nomenclature

C = thrust-specific fuel consumption
C_D = $2D/\rho V^2 S$, drag coefficient, $C_D = C_{D0} + KC_L^2$
C_{D0} = zero-lift drag coefficient
C_L = $2L/\rho V^2 S$, lift coefficient
D = drag force on aircraft
E = $L/D = C_L/C_D$, aerodynamic efficiency of aircraft
h = altitude
K = induced drag factor
L = lift force on aircraft
S = wing planform area
T = thrust required
t = endurance, time taken during the cruise

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V = airspeed
W = total weight, all up weight of aircraft
x = range, horizontal distance covered in the cruise.
ζ = $(W_1 - W_2)/W_1$, cruise-fuel weight fraction
ζ^* = value of ζ for which flight is optimized
ρ = density of atmospheric air
ρ_{SSL} = 1.226 kg/m^3, value of ρ at sea level on standard day
σ = ρ/ρ_{SSL}, density ratio

Subscripts
1 = at the start of the cruise
2 = at the end of the cruise
br = best range
$1br$ = best range at the start of the cruise
m = maximum

Introduction

THIS note develops analytic expressions for the performance calculations in preliminary design phase of the best range cruising flight of a turbojet aircraft. The altitude and airspeed are kept constant during the flight and it is referred to as simply constant h-v flight. This important flight program has been discussed in earlier works[1,2] but, thereafter it seems not to have been developed further, even in recent publications on aircraft performance. We first write the basic equation for the range and, thereafter, the airspeed for maximum range is obtained. This helps to obtain the other best range flight parameters. Finally, this flight is illustrated with a realistic numerical example.

Basic Equation for the Range

The range x between stations 1 and 2 of this constant h-v cruising flight is given[2] as

$$x = -\int_1^2 \frac{V}{CD}\,\mathrm{d}W = -\frac{V}{C}\int_1^2 \frac{C_L}{C_D}\frac{\mathrm{d}W}{W}$$

where C is considered constant. The right side of the above relation can be integrated[2] to obtain the range as

$$x = \frac{2VE_m}{C}\tan^{-1}\left\{\frac{E_1\zeta}{2E_m(1 - KE_1C_{L,1}\zeta)}\right\} \tag{1}$$

where

$$C_{L,1} = \frac{2(W_1/S)}{\rho V^2},\quad E_m = \frac{1}{2(KC_{D0})^{1/2}},\quad \zeta = \frac{W_1 - W_2}{W_1} \tag{2}$$

$$E_1 = \frac{C_{L,1}}{C_{D,1}} = \frac{C_{L,1}}{C_{D0} + KC_{L,1}^2} = \frac{2(W_1/S)\rho V^2}{C_{D0} + 4K(W_1/S)^2/\rho^2V^4} \tag{3}$$

It has been assumed that the aircraft weight during the cruise changes due to fuel consumption only.

Best Range Airspeed

It would be possible to obtain an analytical expression for the best range airspeed V_{br} if we note that the arc tan function in Eq. (1) is much smaller than unity in practice. Therefore, the arc tan function can be approximated by

$$\tan^{-1}\left\{\frac{E_1\zeta}{2E_m(1 - KE_1C_{L,1}\zeta)}\right\} \simeq \frac{E_1\zeta}{2E_m(1 - KE_1C_{L,1}\zeta)} \tag{4}$$

This simplifies Eq. (1) as

$$x = \frac{2(W_1/S)^2\rho\zeta V^3}{C[C_{D0}\rho^2V^4 + 4K(1 - \zeta)(W_1/S)^2]} \tag{5}$$

In order to optimize x with respect to V, it is necessary that all other quantities except V on the right side of Eq. (5) be regarded as constant. Therefore, a fixed value of ζ is taken, denoted as $\zeta = \zeta^*$, for which the flight is optimized. The ζ^* is usually the value of ζ at the end of the cruise. Equation (5) becomes

$$x = \frac{2(W_1/S)^2\rho\zeta^*V^3}{C[C_{D0}\rho^2V^4 + 4K(1 - \zeta^*)(W_1/S)^2]} \tag{6}$$

The above relation is differentiated with respect to V, and taking $\mathrm{d}x/\mathrm{d}V = 0$, the resulting equation is solved to give the best range airspeed V_{br} as

$$V_{br} = \left\{\frac{2(W_1/S)}{\rho_{SSL}\sigma}\right\}^{1/2}\left\{\frac{3K(1 - \zeta^*)}{C_{D0}}\right\}^{1/4} \tag{7}$$

It should be noted that V_{br} depends on ζ^*. This essentially means that the quantity of fuel consumed during the specified cruising flight must be known beforehand, to set for the best range airspeed. The V_{br} decreases with the increase in ζ^*.

Lift Coefficient, Aerodynamic Efficiency, and Thrust

The other flight parameters for the best range would now be obtained. We will regard station 2 as variable by dropping the suffix 2. It should be remembered that the flight under consideration involves constant altitude (and thus ρ) and airspeed. The lift coefficient $C_{L,br}$ and the aerodynamic efficiency E_{br} for the best range would be given by

$$C_{L,br} = (1 - \zeta)\left\{\frac{C_{D0}}{3K(1 - \zeta^*)}\right\}^{1/2} \tag{8}$$

and

$$E_{br} = \frac{C_{L,br}}{C_{D0} + KC_{L,br}^2} = \frac{2E_m(1 - \zeta)\sqrt{3(1 - \zeta^*)}}{3(1 - \zeta^*) + (1 - \zeta)^2} \tag{9}$$

During the cruise, $L = W$ and $T = D$. This gives $T/W = 1/E$. It, therefore, follows that:

$$\frac{T_{br}}{W} = \frac{1}{E_{br}} = \frac{3(1 - \zeta^*) + (1 - \zeta)^2}{2E_m(1 - \zeta)\sqrt{3(1 - \zeta^*)}} \tag{10}$$

From Eqs. (8–10) it can be seen that $C_{L,br}$, E_{br}, and T_{br}/W vary during the flight. At the start of the cruise their values $C_{L,1br}$, E_{1br}, and T_{1br}/W respectively can be obtained by putting $\zeta = 0$ in the Eqs. (8–10) above. The flight parameters at the start of the cruise, therefore, also depend on ζ^* for the best range.

Best Range and Endurance

We obtain the best range x_{br} from relation (1); it can be noted, however, that within the framework of approximation made by relation (4), we can use the relation (5) for obtaining the range as well. Eq. (1) for the best range can be written as

$$x_{br} = \frac{2V_{br}E_m}{C}\tan^{-1}\left\{\frac{E_{1br}\zeta}{2E_m(1 - KE_{1br}C_{L,1br}\zeta)}\right\}$$

which, after using Eqs. (7–9), becomes

$$x_{br} = \frac{2E_m}{C}\left\{\frac{2(W_1/S)}{\rho_{SSL}\sigma}\right\}^{1/2}\left\{\frac{3K(1 - \zeta^*)}{C_{D0}}\right\}^{1/4} \tan^{-1}\left\{\frac{\zeta\sqrt{3(1 - \zeta^*)}}{(1 - \zeta) + 3(1 - \zeta^*)}\right\} \tag{11}$$

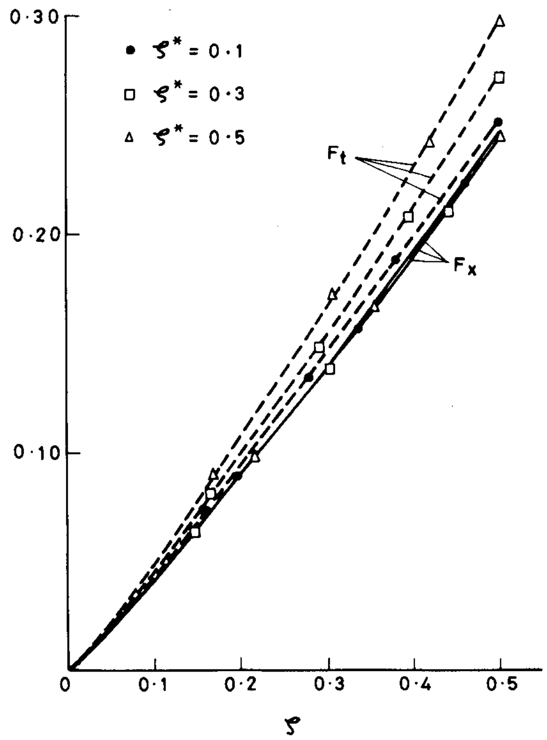

Fig. 1 Changes in F_x and F_t with ζ showing the influence of ζ^*.

Because the airspeed is constant, the endurance t_{br} can be obtained easily as $t_{br} = x_{br}/V_{br}$, which, after using Eqs. (7) and (11) becomes

$$t_{br} = \frac{2E_m}{C} \tan^{-1}\left\{\frac{\zeta\sqrt{3(1-\zeta^*)}}{(1-\zeta) + 3(1-\zeta^*)}\right\} \quad (12)$$

The above two relations (11) and (12), respectively, show the variation of the range and endurance with ζ for the best range flight optimized for $\zeta = \zeta^*$.

It would be pertinent to investigate the effectiveness of ζ^* in Eqs. (11 and 12). For this purpose, the two functions F_x and F_t are defined as

$$F_x = (1-\zeta^*)^{1/4} F_t, \quad F_t = \tan^{-1}\left\{\frac{\zeta\sqrt{3(1-\zeta^*)}}{(1-\zeta) + 3(1-\zeta^*)}\right\}$$

These two functions appear in the expressions for range and endurance, respectively. They are plotted against ζ in Fig. 1 for three different values of ζ^* (0.1, 0.3, and 0.5). The three solid lines of F_x are so close that it becomes difficult to distinguish them; whereas, the three dashed lines of F_t are quite distinct. Therefore, we can state that, unlike endurance, the range is not appreciably affected by ζ^* for a given aircraft.

If we desire to obtain the best range flight parameters for all values of ζ, we put $\zeta^* = \zeta$ in Eqs. (7–12), which give, respectively

$$V_{br} = \left\{\frac{2(W_1/S)}{\rho_{SSL}\sigma}\right\}^{1/2}\left\{\frac{3K(1-\zeta)}{C_{D0}}\right\}^{1/4} \quad (13)$$

$$C_{Lbr} = \left\{\frac{C_{D0}(1-\zeta)}{3K}\right\}^{1/2}, \quad E_{br} = \frac{2E_m\sqrt{3(1-\zeta)}}{4-\zeta} \quad (14)$$

$$\frac{T_{br}}{W} = \frac{4-\zeta}{2E_m\sqrt{3(1-\zeta)}} \quad (15)$$

$$x_{br} = \frac{2E_m}{C}\left\{\frac{2(W_1/S)}{\rho_{SSL}\sigma}\right\}^{1/2}\left\{\frac{3K(1-\zeta)}{C_{D0}}\right\}^{1/4} \tan^{-1}\left\{\frac{0.433\zeta}{\sqrt{1-\zeta}}\right\} \quad (16)$$

and

$$t_{br} = \frac{2E_m}{C}\tan^{-1}\left\{\frac{0.433\zeta}{\sqrt{1-\zeta}}\right\} \quad (17)$$

In the above relations (13–17), each change in ζ would correspond to a different best range flight of the constant *h*-*v* flight program of the given aircraft.

Application to an Aircraft

A turbojet aircraft of wing-loading 4000 N/m^2 (83.6 lb/ft^2) is flying at an altitude of 9 km ($\sigma = 0.380$). It consumes the fuel weight fraction of 0.3 during the cruising flight. It has parabolic drag polar $C_D = 0.016 + 0.052\,C_L^2$ and the thrust specific fuel consumption of 0.8/h. The best range airspeed, lift coefficient, aerodynamic efficiency, thrust-to-weight ratio, range, and endurance of the aircraft can be obtained, using the relations developed in this paper, for standard atmospheric conditions.

The best range airspeed is obtained from Eq. (7) as

$$V_{br} = 211.8 \text{ m/s} = 762.5 \text{ km/h}$$

The variation of the lift coefficient, aerodynamic efficiency, and thrust-to-weight ratio required to get the optimum range are obtained from Eqs. (8–10) as

$$\frac{C_{L,br}}{C_{L,1br}} = 1 - \zeta = \frac{W}{W_1} = \frac{(W/S)}{(W_1/S)},$$

$$\frac{E_{br}}{E_{1br}} = \frac{1-\zeta}{1 + 0.322\zeta(\zeta - 2)},$$

$$\frac{T_{br}}{T_{1br}} = 1 + 0.322\zeta(\zeta - 2)$$

where

$$C_{L,1br} = 0.383, \quad E_{1br} = 16.17, \quad T_{1br}/W_1 = 0.062$$

The variations of these flight parameters with ζ are plotted in Fig. 2. The airspeed and altitude remain constant as per

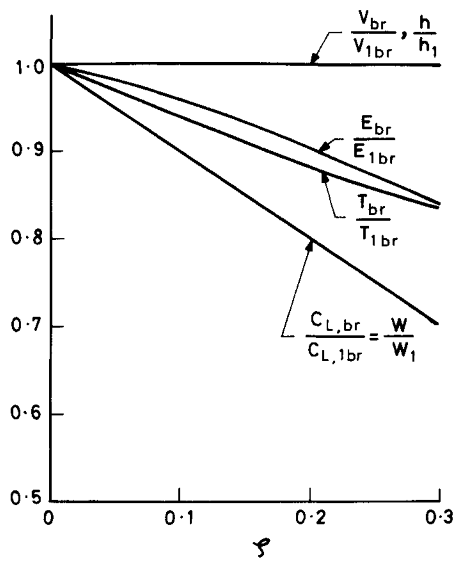

Fig. 2 Variations of best range flight parameters during cruise.

the flight specification. The $C_{L,br}$ varies linearly; whereas, E_{br} and T_{br} vary nonlinearly. All these flight parameters drop during the cruise.

The best range and its endurance are obtained from Eqs. (16) and (17) by putting $\zeta = 0.3$, giving $x_{br} = 5078$ km, and $t_{br} = 6.66$ h.

The aircraft, therefore, flies at the best range airspeed of 762.5 km/h and covers the range of 5078 km in 6.66 h.

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Statistical Prediction of Maximum Buffet Loads on the F/A-18 Vertical Fin

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I. Introduction

IN a previous wind-tunnel investigation[1] of tail buffeting on the F/A-18, the vertical fin normal force was evaluated from a large number of pressure measurements on the fin surfaces. Steady and rms normal force coefficients were computed. For structural integrity considerations, it is of interest to know the peak load and frequency of occurrence. Some wind tunnels, such as the blowdown type used in Ref. 1, can only be operated for short running times. Similarly, in flight tests,[2] data collected within a certain angle of attack and dynamic pressure band are usually of short duration. A reliable statistical method to estimate maximum load encountered for long operating time of the aircraft using short duration experimental data is highly desirable.

A number of statistical approaches have been used successfully to predict maximum instantaneous distortion patterns of engine inlet total pressure. One that is especially suitable for predicting maximum buffet load is that of extreme value statistics, developed by Gumbel[3] and used by Jacocks[4,5] in inlet distortion studies. In this study, results using Gumbel's first and third asymptotic distributions are presented.

II. Gumbel's Extreme Value Statistics

In this section, steps of the procedure to apply Gumbel's asymptotic theory of extreme values to predict buffet loads are given.

If $X_1, X_2, X_3, \ldots, X_n$ denote n independent observations for the same parent population, and x is the maximum of X_i, Gumbel[3] gave three asymptotic distributions of x for large n. These can be expressed by a single equation given by the following generalized asymptotic initial distribution:

$$F(x) = \exp - \left[\frac{\alpha - \beta x}{\alpha - \beta \nu}\right]^{1/\beta} \tag{1}$$

where $\beta = 0$ corresponds to the first asymptote, $\beta < 0$ is the second asymptote, and $\beta > 0$ is the third asymptote. The ratio α/β is the maximum extreme value achievable, ν is the most frequently occurring extreme value level, and α represents the rate of increase of the extreme level with logarithm of time. Gumbel defined two variables; namely, the reduced variate t given as

$$t = -\ln \ln[1/F(x)] \tag{2}$$

and a return period T, which represents the operation time required to observe an extreme equal to or greater than x given by

$$T = 1/[1 - F(x)] = 1/\{1 - \exp[-\exp(-t)]\} \tag{3}$$

The parameters α, β, and ν can be estimated using the method of maximum likelihood. If the parent or initial distribution has a probability density function $f(X_i, \alpha, \beta, \nu)$, the likelihood function can be written as

$$L = \prod_{i=1}^{n} f(X_i, \alpha, \beta, \nu) = \prod_{i=1}^{n} \mathrm{d}[F(X_i, \alpha, \beta, \nu)]/\mathrm{d}x \tag{4}$$

The maximum likelihood estimates of α, β, and ν are obtained from the following equations:

$$\partial \ln L/\partial\alpha = H_1 = 0, \qquad \partial \ln L/\partial\beta = H_2 = 0,$$
$$\partial \ln L/\partial\nu = H_3 = 0 \tag{5}$$

which can be solved using a modified Gauss-Newton iteration scheme.

Let $\theta_1, \theta_2, \theta_3$ represent α, β, and ν and $\theta = \theta(\theta_1, \theta_2, \theta_3)^T$, $H(\theta) = H(H_1, H_2, H_3)^T$. A function $Q(\theta)$ is defined as

$$Q(\theta) = H^T(\theta)H(\theta) = \sum_{i=1}^{3} H_i^2(\theta) \tag{6}$$

$H(\theta)$ can be expanded in a Taylor series about the initial value θ_0 as follows:

$$H(\theta) = H(\theta_0) + H'_\theta(\theta_0)\Delta\theta_0 \tag{7}$$

where

$$H'_\theta = (\partial H_i/\partial\theta_j) \quad \text{for} \quad i = 1, 2, 3; \qquad j = 1, 2, 3 \tag{8}$$

and $\Delta\theta_0 = \theta - \theta_0$. The function $Q(\theta)$ is minimized under the condition

$$\operatorname{grad} Q(\theta) = \operatorname{grad} H^T(\theta)H(\theta) = 0 \tag{9}$$

and $\Delta\theta_0$ is obtained by solving the following equation:

$$H'^T_\theta(\theta_0)H'_\theta(\theta_0)\Delta\theta_0 = -H'^T_\theta(\theta_0)H(\theta_2) \tag{10}$$

Consider the function

$$Q(v) = Q(\theta_k + v\Delta\theta_k) \quad \text{for} \quad 0 \le v \le 1 \tag{11}$$

where the subscript k denotes the kth iteration. Let V_m be the value of v when $Q(v)$ is a minimum in the interval $0 \le v$

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